Physics 222: Solutions for Written Problems in Exam #1

Problem 1: An electron (m=9.1×10^{-31} kg, q=$-e$=-1.6×10^{-19} C) of velocity $\vec{v} = v_x\hat{i} + v_z\hat{k}$ enters a region of uniform magnetic field $\vec{B} = B_z\hat{k}$. Its trajectory will therefore be a helix.

   A. Starting from Newton's second law, show that the radius of the helix is $r = mv_x/(eB)$. (4 pts.)

Solution: The magnetic force is calculated as follows:

$$\vec{F}_{\text{mag}} = q\vec{v} \times \vec{B} = -e\left(v_x\hat{i} + v_z\hat{k}\right) \times B_z\vec{k} = ev_xB_z\hat{j}.$$

Newton's second law ($\vec{F} = m\vec{a}$, $F = ma$) needs to be applied to uniform circular motion in this case, therefore the magnetic force is equal to the mass of the electron times the centripetal acceleration. Since the circular part of the motion is in the xy-plane (the pitch of the helix is along z), only the x- and y-components of the velocity are affected by the centripetal acceleration, which is therefore $a = v_x^2/r$. Newton's second law therefore yields

$$ev_xB_z = mv_x^2/r,$$

which can be solved for r:

$$r = \frac{mv_x}{eB}.$$

   B. Show that the period of the circular component of its motion is $T = 2\pi m/(eB)$. (4 pts.)

Solution: The period is given by the circumference $2\pi r$ divided by the x-component of the velocity. The radius is taken from the result of part A.

$$T = \frac{2\pi r}{v_x} = \frac{2\pi mv_x}{eBv_x} = \frac{2\pi m}{eB}.$$

   C. Show that the pitch of the helix, i.e., the displacement of the electron in the direction of the field lines in one period, is $d = 2\pi mv_z/(eB)$. (2 pts.)

Solution: The pitch of the helix is the z-component of the velocity times the period of the circular component of its motion. The period is taken from part B. above.

$$d = v_zT = \frac{2\pi mv_z}{eB}.$$

Problem 2:   A. Write down the Biot-Savart law for the magnetic field created by a current element. Draw an appropriate figure and identify each quantity in the law.

Solution: You can either write down Biot-Savart's law for a current element (short piece of wire) in its vector form

$$d\vec{B} = \frac{\mu_0}{4\pi}I\frac{\vec{dl} \times \hat{r}}{r^2}$$

or in its scalar form

$$dB = \frac{\mu_0}{4\pi}I\frac{dl\sin\theta}{r^2}$$

The meaning of each quantity is as follows:

- μ_0 is the permeability constant $4\pi \times 10^{-7}$ N/A^2.
- π is $3.14\ldots$.
- $\vec{dB}$ is the magnetic field created by the current element. dB is its magnitude.
- The magnitude of $\vec{dl}$ is the length of the current element; it points in the direction of the current flow.
- I is the current flowing through the current element.
- r is the distance between the current element and the position where the magnetic field is calculated. $\hat{r}$ is a unit vector pointing from the current element to this position.
- θ is the angle between $\vec{dl}$ and $\hat{r}$.

Benson, Fig. 30.5 (b) is the appropriate figure.

B. Using either the Biot-Savart law or Ampere's law (specify which), show that the field B at a distance r from an infinite straight wire carrying a current I is $\mu_0 I / (2\pi r)$. Describe or draw the magnetic field lines. (4 pts.)

Solution: This problem can be solved either with the Biot-Savart law (see Benson, Example 30.2, the lecture notes for section 30 on page 11, or my handout on the Biot-Savart law) or with Ampere's law. Using Ampere's law is much simpler, and I use it here in this solution. Note, however, that Ampere's law does not give you the direction of the magnetic field. The direction of the field is only indicated in the vector form of the Biot-Savart law.

The vector form of the Biot-Savart law states that $\vec{dB}$ is proportional to $\vec{dl} \times \hat{r}$, therefore the magnetic field lines circle the long straight wire in a direction given by the right hand rule. See Benson, Fig. 30.5 (b) for a figure. Because of the cylindrical symmetry of the problem, the field strength can only depend on the distance from the wire.

Ampere's law states that the integral over $\vec{B} \cdot \vec{dl}$ for any closed loop is equal to μ_0 times the enclosed current I. (In this case, $\vec{dl}$ is not a current element, but rather a line element of the integration path.) Choose the loop to be a circle with radius r in a plane perpendicular to the wire, with the wire passing through the center of the circle. Since the magnetic field circles the wire ($\vec{B} \parallel \vec{dl}$), we have $\vec{B} \cdot \vec{dl} = Bdl$. Since the field strength does not depend on the position on the circle, we have

$$\mu_0 I = \int \vec{B} \cdot \vec{dl} = \int Bdl = B \int dl = B 2\pi r.$$

Therefore, the field strength is

$$B = \frac{\mu_0 I}{2\pi r}.$$

B. Using either the Biot-Savart law or Ampere's law (specify which), show that the field B at the center of a current-carrying ring of radius a is $\mu_0 I / (2a)$. (4 pts.)

Solution: This problem can only be solved with the Biot-Savart law (not with Ampere's law). The Biot-Savart law in its scalar form states that

$$dB = \frac{\mu_0}{4\pi} I \frac{dl \sin\theta}{r^2}.$$

We need to plug in the various terms and then integrate. The distance r between the center of the wire and the current element dl (located on the ring) is equal to the radius a. The vector $\hat{r}$ pointing from the current element $\vec{dl}$ to the center of the wire is at a right angle with the current element $\vec{dl}$, therefore $\theta = 90°$. The Biot-Savart law therefore takes the form

$$dB = \frac{\mu_0}{4\pi} I \frac{dl}{a^2},$$

since $\sin\theta = 1$. We need to add the contributions from all current elements in the ring, i.e., integrate over the ring. In this integral, only dl needs to remain inside the integral, since all other quantities are constants:

$$B = \int dB = \int \frac{\mu_0}{4\pi} I \frac{dl}{a^2} = \frac{\mu_0}{4\pi} \frac{I}{a^2} \int dl.$$

The integral $\int dl$ is simply the length of the ring, that is its circumference $2\pi a$. Therefore, the field strength is

$$B = \frac{\mu_0}{4\pi} \frac{I}{a^2} \int dl = \frac{\mu_0}{4\pi} \frac{I}{a^2} 2\pi a = \frac{\mu_0}{2} \frac{I}{a} = \frac{\mu_0 I}{2a}.$$

Problem 3: In the circuit at right, the source is an antenna. The circuit is tuned, by varying the capacitance, to receive WOI AM at f=640 kHz ($\omega = 2\pi f = 4.02 \times 10^6$ rad/s). The amplitude of the source voltage is V_0=5 mV; R=1 kΩ and L=2 mH.

A. Show that the value of C at the point at which the radio is tuned to WOI AM is 31 pF. What is the phase angle ϕ ? (3 pts.)

Solution: In order for the circuit to be at resonance, the impedance needs to be purely Ohmic, i.e., the capacitive and inductive reactances X_C and X_L need to be equal to each other:

$$X_L = \omega L = X_C = \frac{1}{\omega C}.$$

This implies that

$$C = \frac{1}{\omega^2 L} = \frac{1}{(4.02 \times 10^6 \text{ rad/s})^2 \times 2 \text{ mH}} = 31 \text{ pF}$$

The phase angle ϕ between current and voltage is zero at resonance. Don't trust me ? Well, the phase angle is given by

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{\omega L - \frac{1}{\omega C}}{R}$$

$$\tan\phi = \left(4.02 \times 10^6 \text{ rad/s} \times 2 \text{ mH} - \frac{1}{4.02 \times 10^6 \text{ rad/s} \times 31 \text{ pF}}\right)/1 \text{ k}\Omega$$

$$\tan\phi = (8.0 \text{ k}\Omega - 8.0 \text{ k}\Omega)/1 \text{ k}\Omega = 0.$$

B. Find the average (rms) power delivered by WOI AM, via the receiving antenna, to the circuit. (3 pts.)

Solution: How do we do this: If we knew the rms current I, then the rms power is $P = IV$, since the power factor is unity. We find the rms current I by dividing the rms voltage V=5 mV/$\sqrt{2}$=3.54 mV by the impedance

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = R.$$

Therefore, I=3.54 mV/1 kΩ=3.54 μA. The rms power $P = IV \cos\phi$ is P=3.54 mV$\times$3.54 μA=12.5 nW. (This is somewhat low; a realistic radio probably would need a somewhat higher power to operate.)

C. The capacitance is then increased to 34 pF. Find the new phase angle ϕ of the circuit. (2 pts.)

Solution: The phase angle ϕ is given by

$$\tan\phi = (X_L - X_C)/R = \left(\omega L - \frac{1}{\omega C}\right)/R.$$

$$\tan\phi = \left(4.02 \times 10^6 \text{ rad/s} \times 2 \text{ mH} - \frac{1}{4.02 \times 10^6 \text{ rad/s} \times 34 \text{ pF}}\right)/1 \text{ k}\Omega.$$

$$\tan\phi = (8.0 \text{ k}\Omega - 7.3 \text{ k}\Omega)/1 \text{ k}\Omega = 0.7.$$

The phase angle ϕ is therefore ϕ=36°.

D. Is the average power dissipated by the circuit decreased, increased, or remained the same? Explain your answer briefly. (2 pts.)

Solution: The power throughput is maximal at resonance. By detuning the capacitor, we are away from the resonance, and therefore the power dissipated by the circuit has decreased. One could also argue that both the rms current (away from resonance) and the power factor have decreased, whereas the rms voltage remained the same. Therefore, the rms power dissipated by the circuit has decreased.